Variational methods for conservative evolution equations

Yann Brenier[*]

Abstract

Variational methods have been widely used for the mathematical and numerical analysis of static models and are often related to elliptic partial differential equations. Examples include elastostatics, composite materials, liquid crystals, supraconductivity etc... Variational methods also play an important role in the modelling of dissipative mechanisms, through the concept of gradient flows and "minimizing motions". The goal of these lectures is to emphasize the importance of variational methods for conservative systems, such as hyperbolic systems of conservation laws, or inviscid models in fluid mechanics.

First week: space-time variational problems

The Euler equations (1755) describe the motion of inviscid incompressible fluids. They obey a space-time variational principle (least action principle). The relationship between the intial value problem and the space-time minimization problem is discussed.

Second week: systems of hyperbolic conservation laws with variational structure

Many conservative systems enjoy additional conservation laws, due to their variational structure (through Noether's theorem). We show how such

[*]CNRS, FR 2800, Université de Nice, brenier@math.unice.fr

properties can be exploited to establish the well-posedness of these models. The related relative entropy/modulated energy method will be shown to be an efficient and simple tool for asymptotic analysis. Several examples related to electrodynamics will be discussed (Euler-Poisson, Born-Infeld equations).

Third weak: zero-scale dissipation and realistic vanishing viscosity solutions of formally conservative equations

For many conservative systems, solutions develop singularities in finite time It is customary to add artificial viscosity and let it go to zero in order to extend solutions beyond singularities. This usually implies some loss of energy (or increase of entropy) although there is no explicit scale of dissipation. For some specific (pressureless) models in hydrodynamics and magnetohydrodynamics, involving sticky particles, springs or strings, we show that, following variational principle, new vanishing viscosity methods can be designed with more realistic viscous effects.